THE

MATHEMATICAL GAZETTE.

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THE PRINCIPLES OF DYNAMICS.

(Continued from p. 394, Vol. III.)

IX. The expression "The Kinetic Energy of a body."

(1) We have, above, assigned an intelligible meaning to this expression *if the body be considered as a member of a system.* But, taken absolutely, it has no meaning.

(2) If our shot, now considered separately, meet directly a target of mass M' and there be inelastic collision, and if the relative velocity be U, then the velocity v' of the shot relatively to the *c.m.* of this new system will be $\frac{M'+m}{M'}\times U$, and the velocity V' of the target relatively to this *c.m.* will be $\frac{m}{M'+m}\times U$.

The total internal energy of this system as measured by the heat given out will be $[\frac{1}{2}m(v')^2+\frac{1}{2}M'(V')^2]$, and of this the shot may with some reason be said to contribute the first term, and the target the second, in virtue of its K.E. [see VII., (3) and (4)].

(3) So that we cannot in general attach any meaning even to the more guarded expression "the K.E. of a body relatively to another body." Only if the target had relatively infinite mass could the shot be said to have a definite "kinetic energy relatively to the target." More correctly we should say that the internal kinetic energy of the *system* of shot and target may now be expressed in terms of the mass of the shot and the relative velocity of shot with respect to the target, viz. as $\frac{1}{2}mU^2$. For, with M' infinite, we may take v' as equal to U, and may neglect $\frac{1}{2}M'.(V')^2$ in comparison with $\frac{1}{2}m(v')^2$; so that the K.E.

$$[\tfrac{1}{2}m(v')^2+\tfrac{1}{2}M'.(V')^2]$$

reduces, as we have just said, to $\frac{1}{2}mU^2$.

X. The Kinetic Energy of a System. Conservation of Energy.

(1) If we refer the motions of any system to any origin O and "fixed" axes, and if v' gives the velocity of any particle in it relatively to O, and v its velocity relatively to the *c.m.* G of the system, and V the velocity of this *c.m.* G relatively to the origin O, then we know that, by the properties of the centre of mass,

$$\Sigma[\tfrac{1}{2}m(v')^2]=\tfrac{1}{2}[\Sigma(m)]\,.\,V^2+\Sigma[\tfrac{1}{2}mv^2]. \quad \text{.............(i)}$$

When we speak of "the K.E. of the solar system" (*e.g.*) we usually refer to its centre of mass G, and so get the simpler expression

$$\Sigma(\tfrac{1}{2}mv^2). \quad \text{........................(ii)}$$

(2) This last expression, the second term of the more general expression given in (i), is the term affected by internal reactions within the system; while if O were the *c.m.* of a still larger system, the expression given in (i) would represent the contribution of K.E. made by the solar system towards the K.E. of this larger system; the term $\frac{1}{2}[\Sigma(m)]\,.\,V^2$ being unaffected by reactions within the solar system (which is now a sub-system).

(3) Hence it seems best to define "the K.E. of a system," when the system is considered by itself, by the term $\Sigma(\frac{1}{2}mv^2)$, where velocities are referred to the *c.m.* of the system.

(4) *Conservation of Energy.* This law refers to a self-contained system, or one that includes all the bodies between which reactions occur. Part of its energy at any moment will be the visible dynamical kinetic energy referred to above. [In the above we have not intended to include the energy of molecular vibration].

Referred to any origin whatever, this part of the energy of the system takes the form given in (1) (i) above.

(5) When V is constant, the term $\frac{1}{2}[\Sigma(m)]\,.\,V^2$ is constant, it is true; but it has nothing to do with the energy of the system considered by itself.

(6) While if the *c.m.* G of the system have an acceleration relatively to the origin O chosen, this term will in general change its value as time goes on, since V is now not constant.

(7) Hence in applying the law of conservation of energy to any self-contained system we may consider it best to refer all velocities, etc., to the *c.m.* of the system (and to "fixed" axes), rather than to any other origin.

XI. Cases in which there is no need to refer explicitly to "the system."

(1) As explained in V., our first ideas as to the laws of dynamics are derived from experiments, made on the earth's surface, in which we start with the ideas of *force, mass, velocity, direction*, etc., already existing in the mind of what may be

termed "the natural man." But laws based on ideas of velocity, acceleration, and displacement as referred to the earth's surface and to landmarks on it do not suffice when we (*e.g.*) deal with the behaviour of the gyroscope, or of Foucault's pendulum, and still less when we deal with the solar system. So we proceeded to give them a wider basis (see VI.). In a vast number of cases, however, we can, quite consistently with these more general principles, revert to the simpler conceptions of V.; and we can in these cases, without sensible error and with much gain as regards simplicity of treatment, desert *the system* and *stresses*, and consider *single bodies* and *forces*.

(2) If (*e.g.*) we are considering the action between the earth and a stone when the latter falls from the top of a house, the laws of VI. tell us that we can without sensible error refer to the *c.m.* of the earth instead of to the common *c.m.* of stone and earth which practically coincides with the former.

(3) If further we disregard the diurnal rotation, we do not even at the equator make an error of more than about 0·35 per cent. in judging of the pull exerted by the earth on the stone from the observed approach of the latter to the former.

(4) Or, more generally, when dealing with cases in which one mass is relatively infinite, and the rotation relatively to the distant stars (or to "fixed" axes) so small that the force required to deflect the path of the smaller mass is but a small fraction of the whole force with which it is urged towards the large mass, *we may refer velocities, etc., to the surface of the large mass and to landmarks on it.* It is for this reason that our first experiments can give us a very good idea of the more general laws.

(5) *Work and kinetic energy.* So again, referring once more to the simple case of VI., (3) and to IX., (3), we see that in the consideration of work and kinetic energy we may in such cases, without sensible error, *refer the displacement and velocity of the smaller body to the surface of the larger body, and to landmarks on it* when considering the work and kinetic energy done in and appearing in any reaction between the two bodies; *and may consider the kinetic energy of the smaller body only.*

(6) So in all similar cases—as (*e.g.*) the action of steam in causing the piston and other movable parts of an engine to move relatively to the framework of an engine and to the earth to which the framework is attached—we may refer to the earth as fixed.

(7) In the less simple case of a locomotive again, we reckon the work done by the steam by considering its pressure and change of volume, and can measure the corresponding kinetic energy that appears by referring the motion of the whole train and of the smaller parts in it to the earth (regarded as fixed); the

reaction in this case also taking place between the earth, of relatively infinite mass, and a group of bodies of masses that are very small relatively to the earth.

XII. Conclusion.

In conclusion it may not be amiss to re-state briefly some of the points insisted on in the above.

(1) *Origin and axes.* In most text-books no clue is given as to the origin and axes to which displacements, velocities, and accelerations are referred. There is presented to the student a body and force acting on it; and the acceleration of, or work done on, the body is discussed.

But we can conceive of any number of origins in space, having any velocities *or accelerations* relatively to the body; which are we to choose? And what axes are we to regard as fixed? If we begin by taking origin and axes fixed relatively to the earth, it should be clearly stated that we do so, as is done in V., (1). If we take origin and axes for which the laws of dynamics hold good with accuracy, as in VI., (1), this again should be stated clearly.

(2) *Force and Mass.* It should be made clear whether we start with independent measures of force and mass and search by experiment for the laws of dynamics—[the course taken in this paper]—or whether we define force and mass dynamically, and then try by experiment whether measures so obtained agree with what may be called statical measures. It appears to the writer that the student is, as a rule, left in doubt as to whether Newton's laws are definitions or true experimental laws; it is hardly made clear where experiment "comes in."

(3) *Energy.* There is no doubt but that, with specification of origin and axes, a perfectly definite meaning can be assigned to the expression "The Energy of a system."

But, too frequently, the "kinetic energy of a body," or "the work done on a body," is spoken of, without any reference to a system; and, as has been pointed out in this paper, this leads to lack of significance or even to error.

XIII. Some suggestions as to the teaching of dynamics.

(1) Care must of course be taken not to confuse the beginner by presenting to him too many ideas at once. He must learn bit by bit; but he should be given to understand that the first bit of truth is not the whole truth, that it is an approximation to it.

(2) *Kinematics.* In the usual preliminary discussion of kinematics, the essentially *relative* nature of all motion as we know it can be insisted on at once; and axes and origin should be specified. In dealing with the movements on a chess-board, we may take the corner and edges of the board as origin and axes, however the board itself may be moved about.

For movements about in a railway carriage we may refer to the carriage as fixed. For movements of the train, or of ships at sea, to the surface of the earth as fixed. For movements of the moon round the earth we may refer to the centre of the earth as origin, and to lines fixed relatively to the distant stars as axes. And so on. The conception of absolute velocity should be discouraged by continual recurrence to this idea of the relativity of motion.

(3) *Kinetics.* In beginning to discuss forces, masses, and accelerations, etc., it would be better to state plainly that we are going at first to take the surface of the earth as "at rest," in the sense that we are going to refer to an origin and to axes fixed on it; and that, in the problems at first discussed, the choice of such origin and axes is justified by experiment. The writer is of opinion that the line of approach to the laws should be that indicated in V.

The discussion of "circular motion" (stones whirled at the end of a string, etc.) would naturally lead up to the consideration of what is meant in dynamics by "change in direction of motion"; and the dynamical definition of "fixed axes" as "axes fixed with respect to distant stars" should be given as one based on experiment, [see III., (4) and (5)]; it should be understood that the expression "absolute rotation" can have for us no more than a dynamical meaning.

Later on, the consideration of "the system" would be needed for a proper understanding of the principles of *Conservation of Momentum* and of *Conservation of Energy*; and working back from this, the considerations presented in XI. would show the student why, and to what extent, we were justified in referring to origin and axes fixed on the earth in the first discussion of the laws of dynamics and in the earlier (terrestrial) problems considered. In fact a re-consideration of all the earlier problems from the point of view of "the system of reacting bodies, a suitable origin [see VI., (1), (ii)], and fixed axes," would probably add much to the student's grasp of the whole subject.

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THE INTEGRAL CALCULUS THEOREM.

The following presentation of the fundamental theorem of The Integral Calculus has been evolved as the result of attempts to put § 31 of Whittaker's "Analysis" before a University class. It is to be regarded as the first application of the Infinite Sequence Theory which has been the subject of articles in recent numbers

of *The Mathematical Gazette.** It is in the details, making for stricter logical accuracy, that what is new will be found.

The real case, as being of special importance, is alone considered: *If $f(x)$ denote the expression of a continuous function of the real variable x in the range $a \leqq x \leqq b$; if x_r be a real function of the integral variables r and n such that it increases with r when n does not vary and that*

$$\operatorname*{Lt}_{n=\infty} (x_{r+1}-x_r)=0, \ (r=0, 1, 2, \ldots, n-1),$$

related to the range by the equations $x_0=a$, $x_n=b$; and if x_r' be another real function of r and n such that $x_r \leqq x_r' \leqq x_{r+1}$, then $\sum\limits_{r=0}^{r=n-1} (x_{r+1}-x_r)f(x_r')$ tends to a finite limiting value when n tends to infinity, for given finite values of a and b.

It is proved:

(i) that an alteration of the law which determines the function x_r' produces a change that vanishes in the limit:

We use S_n' for the summation, S_n'' for a summation of the same type in which x_r' is replaced by x_r''; if ϵ denotes a small positive number, a positive number (η) can be determined so that

$$|f(x+\theta\eta)-f(x)| < \epsilon, \text{ for all values of } x \text{ in the range,}$$

and a positive integer (N) can be determined so that

$$(x_{r+1}-x_r) < \eta, \ (r=0, 1, 2, \ldots, n-1) \text{ provided that } n > N,$$

and therefore $\quad |f(x_r'')-f(x_r')| < \epsilon$, if $n > N$.

Whence

$$|S_n''-S_n'| < \epsilon(b-a) \text{ if } n > N; \text{ therefore } \operatorname*{Lt}_{n=\infty} (S_n''-S_n')=0.$$

(ii) If $\sum\limits_{0}^{n-1}(x_{r+1}-x_r)f(x_r)$ be denoted by S_n, it need only be proved that S_n has a limit, and that this limit does not depend on the law which specifies x_r.

Let ξ_r denote another function of r and n satisfying the same conditions as x_r, and let Σ_n denote $\sum\limits_{0}^{n-1}(\xi_{r+1}-\xi_r)f(\xi_r)$.

Let ϵ, η have the same meaning as in (i); and let N denote a positive integer such that if $n > N$, both

$$(x_{r+1}-x_r) < \eta \text{ and } (\xi_{r+1}-\xi_r) < \eta, \ (r=0, 1, 2 \ldots, n-1).$$

Let n_1, ν_1 denote two integers greater than N, and let the *numbers*

$$x_1, x_2 \ldots x_{n_1-1}, \ \xi_1, \xi_2, \ldots \xi_{\nu_1-1}$$

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* Cf pp. 329-335, Vol. III., and pp. 357-365; in particular, p. 362.

constitute the *increasing* system $U_1, U_2, \dots U_{k-1}$; k being an integer not greater than $(n_1+\nu_1-1)$.

Then we may clearly write

$$\Sigma_{\nu_1}-S_{n_1}=\sum_{r=0}^{r=k-1}(U_{r+1}-U_r)\{f(\xi_\mu)-f(x_m)\},$$

μ and m depending on r in such a way that

$$x_m \leqq U_r < U_{r+1} \leqq x_{m+1} \text{ and } \xi_\mu \leqq U_r < U_{r+1} \leqq \xi_{\mu+1}.$$

The following is then a diagrammatic representation of the possibilities:

$$\begin{array}{rl}
\text{I.} & x_m, \dots\dots, \xi_\mu(\equiv U_r), x_{m+1}(\equiv U_{r+1}), \dots\dots, \xi_{\mu+1} \\
\text{II.} & x_m, \dots\dots, \xi_\mu(\equiv U_r), \xi_{\mu+1}(\equiv U_{r+1}), \dots\dots, x_{m+1} \\
\text{III.} & \xi_\mu, \dots\dots, x_m(\equiv U_r), \xi_{\mu+1}(\equiv U_{r+1}), \dots\dots, x_{m+1} \\
\text{IV.} & \xi_\mu, \dots\dots, x_m(\equiv U_r), x_{m+1}(\equiv U_{r+1}), \dots\dots, \xi_{\mu+1};
\end{array}$$

the dots representing members of the U-system, all ξ's in II., all x's in IV, first ξ's then x's in I, first x's then ξ's in III.

Thus $\quad |\xi - x_m| < (x_{m+1}-x_m)$ in I. and II.,

and $\quad < (\xi_{\mu+1}-\xi_\mu)$ in III. and IV.;

i.e. in all cases $|\xi_\mu - x_m| < \eta$, and therefore

$$|f(\xi_\mu)-f(x_m)| < \epsilon.$$

[It should be noticed that the possibilities I. and III. require the reduction of the problem that is accomplished in (i) above.]

Hence $\quad |\Sigma_{\nu_1}-S_{n_1}| < \epsilon(b-a)$, and it follows that

$$|\Sigma_\nu - S \quad| < \epsilon(b-a) \text{ if } n, \nu > N.$$

Thus, in particular,

$$|S_{n+p}-S_n| < \epsilon(b-a) \text{ if } n > N, \ p \text{ positive integral,}$$

hence S_n tends to a limit when n tends to infinity;

and again $\quad |\Sigma_n - S_n| < \epsilon(b-a)$ if $n > N$,

i.e. $\quad \underset{n=\infty}{\mathrm{Lt}}(\Sigma_n - S_n) = 0,$

or, the limits towards which Σ_n, S_n tend are identical.

The care with which, throughout these articles, the symbols have been specified as sometimes variables, sometimes numbers (values of the variables), is perhaps worthy of attention; slackness in this respect is the cause of much obscurity and even inaccuracy, and tends to lead students into careless ways of thinking.

D. K. PICKEN.

ON THE EXPONENTIAL INEQUALITIES AND THE EXPONENTIAL FUNCTION.

§ 1. THEOREM. *If a be any positive quantity not equal to* 1, *and x, y, z be any three rational quantities in descending order of magnitude, then*

$$a^x(y-z)+a^y(z-x)+a^z(x-y)>0.$$

Proof. Let p, q, r be any three integers in descending order of magnitude. Since $(a-1)^2>0$, we have $a^2-a>a-1$. Hence in general

$$a^m-a^{m-1}>a^{m-1}-a^{m-2};$$

from which we have,

$$(a^p-a^{p-1})>(a^{p-1}-a^{p-2})>\ldots>(a^{q+1}-a^q)$$
$$>(a^q-a^{q-1})>\ldots>(a^{r+1}-a^r).$$

The arithmetic mean of the terms in the first row should be greater than the arithmetic mean of the terms in the second row, since each of the terms in the first row is greater than each of the terms in the second row. Therefore

$$\frac{a^p-a^q}{p-q}>\frac{a^q-a^r}{q-r},$$

or $$a^p(q-r)+a^q(r-p)+a^r(p-q)>0. \quad\text{(A)}$$

Let n be any positive integer. In the above, for a, we can write $a^{\frac{1}{n}}$. Doing so, and dividing throughout by n, we get

$$a^{\frac{p}{n}}\left(\frac{q}{n}-\frac{r}{n}\right)+a^{\frac{q}{n}}\left(\frac{r}{n}-\frac{p}{n}\right)+a^{\frac{r}{n}}\left(\frac{p}{n}-\frac{q}{n}\right)>0. \quad\text{(B)}$$

(A) proves the theorem when x, y, z are integers, and (B) in any other case.

§ 2. In the theorem of the above section, for $x>y>z$, write

first, $m>1>0$;
secondly, $1>m>0$;
and thirdly, $1>0>m$.

And the result is,

in the first case, $a^m-1>m(a-1)$;
in the second case, $a^m-1<m(a-1)$;
in the third case, $a^m-1>m(a-1)$.

Hence, *if a is positive and not equal to* 1, $a^m-1\gtrless m(a-1)$, according as $m(m-1)\gtrless 0$.

§ 3. In the theorem of the first article, for $x>y>z$, write

first, $x>y>0$;
secondly, $x>0>y$;
thirdly, $0>x>y$.

And in each case the result reduces to $\dfrac{a^x-1}{x}>\dfrac{a^y-1}{y}$. Consequently, *if a is positive and* $\neq 1$, and $x>y$, $\dfrac{a^x-1}{x}>\dfrac{a^y-1}{y}$.

§ 4. Let x and y be two positive quantities, $x\neq y$. In the result of § 2, for a write, first, x/y, and then y/x, and reducing, we get the result,

$$mx^{m-1}(x-y)\gtrless x^m-y^m\gtrless my^{m-1}(x-y)$$

according as $m(m-1)\gtrless 0$.

COR. 1. If $x > y > 0$,

$$mx^{m-1} \gtrless \frac{x^m - y^m}{x-y} \gtrless my^{m-1},$$

according as $m(m-1) \gtrless 0$.

COR. 2. If a is any positive quantity not equal to 1,

$$ma^{m-1}(a-1) \gtrless a^m - 1,$$

according as $m(m-1) \gtrless 0$.

§ 5. Let a and b be positive, $a \neq b$; and let p and q be any two positive rational numbers. In the inequality of § 1, for a write $\frac{a}{b}$; and for $x > y > z$ write $1 > \frac{p}{p+q} > 0$. We get

$$\left(\frac{a}{b}\right)^1\left(\frac{p}{p+q} - 0\right) + \left(\frac{a}{b}\right)^{\frac{p}{p+q}}(0-1) + \left(\frac{a}{b}\right)^0\left(1 - \frac{p}{p+q}\right) > 0,$$

or

$$\frac{a}{b} \cdot \frac{p}{p+q} + \frac{q}{p+q} > \left(\frac{a}{b}\right)^{\frac{p}{p+q}}.$$

Hence

$$\frac{pa+qb}{p+q} > (a^p b^q)^{\frac{1}{p+q}}.$$

COR.

$$\frac{pa+qb+\dots+tk}{p+q+\dots t} > (a^p b^q \dots k^t)^{\frac{1}{p+q+\dots+t}},$$

if $p, q, \dots t$ be positive and rational, and $a, b, \dots k$ positive but not all equal.

§ 6. THEOREM. a^x is a *compact* function of the rational variable x.

Proof. Let p, q, x, y, p', q' be any six rational numbers in descending order of magnitude. Now the inequality in the first section can be written

$$\frac{a^x - a^y}{x-y} > \frac{a^y - a^z}{y-z}, \text{ if } x > y > z.$$

Hence,

$$\frac{a^p - a^q}{p-q} > \frac{a^q - a^x}{q-x} > \frac{a^x - a^y}{x-y} > \frac{a^y - a^{p'}}{y-p'} > \frac{a^{p'} - a^{q'}}{p'-q'}.$$

Suppressing the 2nd and 4th terms in the above relation and multiplying by $x-y$, we get

$$\frac{a^p - a^q}{p-q}(x-y) > a^x - a^y > \frac{a^{p'} - a^{q'}}{p'-q'}(x-y).$$

In this inequality the extreme terms can be made as small as we please by making $x-y$ small enough. Hence the middle term $a^x - a^y$ can also be made as small as we please by making $x-y$ sufficiently small; which proves the theorem.

A simpler inequality suited to the same purpose can now be stated; that is, if $p+1, p, x, y, q+1, q$ be in descending order of magnitude,

$$(a^{p+1} - a^p)(x-y) > a^x - a^y > (a^{q+1} - a^q)(x-y),$$

a being any positive quantity $\neq 1$.

§ 7. From the above inequality we could define a^x for irrational values of x and establish the Theory of Irrational Indices; that is, extend the Theory of Indices to irrational values, or to real values in general, of the exponents.

Thus completely defined, a^x becomes a *continuous* function of x, which, if $a>1$, increases from 0 to $+\infty$ as x increases from $-\infty$ to $+\infty$, and if $a<1$, decreases from $+\infty$ to 0 as x increases from $-\infty$ to $+\infty$.*

§ 8. It follows from the above that every positive quantity x has a definite logarithm with respect to any base a, if a is positive and not equal to 1. It follows further that if $a>1$, $\log a^x$ increases from $-\infty$ to $+\infty$ as x increases from 0 to ∞; and if $a<1$, $\log_a x$ decreases from $+\infty$ to $-\infty$ as x increases from 0 to ∞.

§ 9. In this section we shall consider the graph of the function $\dfrac{a^x-1}{x}$ as x decreases from a positive to a negative value.

We have already observed that a^x is a continuous function of x. Hence, by the Theory of Continuity, $\dfrac{a^x-1}{x}$ is a continuous function of x. From the inequality of § 3, $\dfrac{a^x-1}{x}$ decreases constantly as x decreases. Hence we see that the graph is a Continuous Curve, with a sudden decrease of value, possibly, as x passes through the value 0 at which $\dfrac{a^x-1}{x}$ becomes indeterminate. The identity $\dfrac{a^x-1}{x}\equiv a^x.\dfrac{a^{-x}-1}{-x}$, however, shows that there is no sudden change of value as x passes through the value 0, since $\underset{x=0}{\mathrm{L}}\ a^x=1$. Hence it follows that $\dfrac{a^x-1}{x}$ has a definite finite limit when $x=0$. We shall represent this limit by λ and shall determine its value presently. We notice that λ should lie between the values of $\dfrac{a^x-1}{x}$, when $x=+1$, and when $x=-1$; that is, between $a-1$ and $1-\dfrac{1}{a}$. We further notice that if x is numerically less than 1, $\dfrac{a^x-1}{x}$ lies between $a-1$ and $1-\dfrac{1}{a}$.

§ 10. We now proceed to give a demonstration of the Exponential Theorem. We have

$$a^x=\{1+(a^{\frac{x}{n}}-1)\}^n,\ n \text{ being any positive integer,}$$

$$=\left\{1+\frac{x}{n}\left(\frac{a^{\frac{x}{n}}-1}{\frac{x}{n}}\right)\right\}^n$$

$$=\left(1+\frac{x}{n}.\lambda_n\right)^n \text{ where } \lambda_n=\frac{a^{\frac{x}{n}}-1}{\frac{x}{n}}$$

$$=1+X_1+X_2+X_3+\ldots+X_n \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\text{where } X_r=\frac{n(n-1)\ldots(n-r+1)}{\lfloor r}.\left(\frac{x}{n}\right)^r.\lambda^r{}_n$$

$$=\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{r-1}{n}\right).\frac{x^r}{\lfloor r}.\lambda^r{}_n.\ \ldots\ldots(2)$$

* It may be observed that the extreme values of a^x follow as follows: If $x>1$, we have $a^x>1+x(a-1)$. Consequently, if $a>1$, $\underset{x=\infty}{\mathrm{L}}\ a^x=\infty$. Hence $\underset{x=-\infty}{\mathrm{L}}\ a^x=0$, if $a>1$; $\underset{x=\infty}{\mathrm{L}}\ a^x=0$, if $a<1$; and $\underset{x=-\infty}{\mathrm{L}}\ a^x=\infty$, if $a<1$.

We shall suppose n numerically greater than x to start with; so that $\frac{x}{n}$ is numerically less than 1. It follows that λ_n or $\dfrac{a^{\frac{x}{n}}-1}{\frac{x}{n}}$ lies between $a-1$ and $1-\frac{1}{a}$, however much n be hereafter increased. Choose any positive quantity κ so as to be numerically greater than both $a-1$ and $1-\frac{1}{a}$. Using modular notation for numerical values, it follows that $|\lambda_n|$ or $\left|\dfrac{a^{\frac{x}{n}}-1}{\frac{x}{n}}\right|$ is always less than κ. Hence from (2),

$$|X_r| < \frac{x'^r}{\lfloor r}.\kappa^r \text{ where } x'=|x|. \quad \text{..........................(3)}$$

From this we see that each term in (1) after the first is numerically less than the corresponding term in the series $1+\dfrac{x'\kappa}{\lfloor 1}+\dfrac{x'^2\kappa^2}{\lfloor 2}+$......................(4)

This is a convergent series of positive terms.

We shall denote by R_p the remainder after p terms of this series. Going back to (1), we shall denote by $R'_{p,n}$ the remainder after the first p terms therein, that is $R'_{p,n}\equiv X_p+X_{p+1}+\ldots+X_n$. It follows that $|R'_{p,n}| \ngtr R_p$, however great n may become, since the numerical value of each of the terms constituting $R'_{p,n}$ is less than that of the corresponding term in R_p, in which the terms are all positive. We thus have

$$a^x=1+X_1+X_2+\ldots+X_{p-1}+R'_{p,n} \quad \text{.........................(5)}$$

where $R'_{p,n}$ is numerically not greater than R_p, however great n may be.

Now making n infinite, we see from (2) that

$$\underset{n=\infty}{\mathrm{L}} X_r=\frac{x^r}{\lfloor r}.\lambda^r \text{ where } \lambda=\underset{x=0}{\mathrm{L}}\frac{a^x-1}{x}.$$

We thus obtain

$$a^x=1+\frac{x\lambda}{\lfloor 1}+\frac{x^2\lambda^2}{\lfloor 2}+\ldots+\frac{x^{p-1}\lambda^{p-1}}{\lfloor p-1}+R'_p$$

where $|R'_p|$ is less than R_p, the remainder after p terms in the series

$$1+\frac{x'\kappa}{\lfloor 1}+\frac{x'^2\kappa^2}{\lfloor 2}+\ldots \quad \text{.................................(6)}$$

Here p may be any integer whatever. If we increase p, R_p steadily decreases and becomes infinitely small as p becomes infinitely great. Therefore also R'_p becomes infinitely small as p becomes infinitely great. That is, we obtain the expansion

$$a^x=1+\frac{x\lambda}{\lfloor 1}+\frac{x^2\lambda^2}{\lfloor 2}+\ldots\text{(to infinity) where } \lambda=\underset{x=0}{\mathrm{L}}\frac{a^x-1}{x}. \quad \text{.........(7)}$$

Writing in particular $\frac{1}{\lambda}$ for x, we get

$$a^{\frac{1}{\lambda}}=1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\ldots=e; \quad \therefore\ a=e^\lambda \text{ or } \lambda=\log_e a. \quad \text{................(8)}$$

We thus find that $\underset{x=0}{\mathrm{L}} \dfrac{a^x - 1}{x} = \log_e a$, and get the expansion

$$a^x = 1 + \frac{x \log a}{\lfloor 1} + \frac{x^2 (\log a)^2}{\lfloor 2} + \dots \qquad \text{.........................(9)}$$

In particular,

$$e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \dots \qquad \text{................................(10)}$$

§ 11. Since $\underset{x=0}{\mathrm{L}} \dfrac{a^x - 1}{x} = \log_e a$, it follows from the discussion in § 9, that *if x be any positive quantity*

$$\frac{a^x - 1}{x} > \log a > \frac{a^{-x} - 1}{-x}.$$

Cor. 1. $$a - 1 > \log a > 1 - \frac{1}{a}.$$

Cor. 2. $$\frac{e^x - 1}{x} > 1 > \frac{e^{-x} - 1}{-x};$$

or *if x is any positive quantity*

$$e^x > 1 + x; \text{ and } e^{-x} > 1 - x.$$

Cor. 4. If $x > y$,

$$a^x \log a > \frac{a^x - a^y}{x - y} > a^y \log a.$$

For, $x - y$ is positive, and therefore,

$$\frac{a^{x-y} - 1}{x - y} > \log a > \frac{a^{-(x-y)} - 1}{-(x - y)}$$

which reduces to the above.

Cor. 5. If $x > y$,

$$e^x > \frac{e^x - e^y}{x - y} > e^y.$$

Cor. 6. If $x > y > 0$,

$$\frac{1}{x} < \frac{\log x - \log y}{x - y} < \frac{1}{y}.$$

For, since $x > y > 0$, and $e > 1$, $\log x > \log y$.

Substituting $\log x$, $\log y$ for x, y in Cor. 5, we get

$$x > \frac{x - y}{\log x - \log y} > y;$$

whence, by taking reciprocals, we get the Logarithmic Inequality above.

It will be observed that Corollaries 4, 5, and 6 give the differential coefficients of a^x, e^x and $\log x$ even as the Power Inequality (Cor. 1, § 4) gives the differential coefficient of x^m.

I hope these notes will be found useful by readers, and should this be the case, and I be allowed some further space in the *Gazette*, I hope to give some additional notes.

Gooty, India,
25th June, 1906.

V. Ramaswami Aiyar.

MATHEMATICAL NOTES.

223. [D. 6. b.] *Higher Trigonometry.*

Since I wrote my rather fragmentary 'Notes' on Higher Trigonometry I have had occasion to work out the theory of the elementary transcendental functions in a rather more systematic way. This attempt has led me to modify my views in some respects. The net result is that I disagree with Mr. Picken more decidedly than I should have done if I could have seen his method of developing the theory six months ago.

I have no time to discuss the question at length. My chief difference with Mr. Picken is about the use to be made of the theorem

$$\lim_{n=\infty}\left(1+\frac{x}{n}\right)^n=\exp x.$$

He makes it fundamental: in fact he takes it as his definition of the exponential function. I adhere to my statement that this is 'logically quite wrong.' Of course I do not mean by this that it is impossible to base a rigorous theory of $\exp x$ and $\log x$ upon this theorem: many writers have done so. What I mean is that to do so is to disturb the proper perspective of the subject. Lewis Carroll based a theory of parallels on the proposition, 'In every Circle, the inscribed equilateral Tetragon is greater than any one of the Segments which lie outside it.' He would have been the first to admit that this was, 'although possible, logically quite wrong.'

Moreover, the result is not encouraging. It is 'not for the immature schoolboy mind.' I am sanguine enough to believe, on the other hand, that it is quite possible for a clever schoolboy to master a good deal of the theory of these functions. But we must look about for methods other than Mr. Picken's. On the whole I incline to the integral definition of $\log x$ as the best starting point. Mr. Picken, I notice, in one place seems to presuppose this definition. If so, why not define the exponential as the inverse of the logarithm? From the equations

$$y=\int_1^x \frac{dt}{t},\quad x=\exp y$$

the greater part of the theory follows with perfect rigour and extreme simplicity. In particular the theorem

$$\lim_{x=\infty}\left(1+\frac{1}{x}\right)^x=e$$

(which Mr. Picken, to judge by his remarks at the top of p. 360, seems to find rather a stumbling block) follows in two or three lines.

There is, of course, really no difference between starting from these definitions and starting from

$$\frac{dx}{dy}=x,\quad x_{y=0}=1$$

as the definition of $x=\exp y$. In this case the logarithm is defined as an inverse function.

Otherwise it seems to me, in spite of what Mr. Picken says, that we must start from the exponential series. The only serious difficulty is the proof of the functional equation by multiplication of series, or the proof of the equation $\frac{dx}{dy}=x$ by differentiation of an infinite series. It is not necessary to face both difficulties: either may be used to avoid the other. Mr. Picken's criticisms (pp. 332-3) I cannot altogether follow.

May I make two other remarks? (1) How does Mr. Picken's investigation of the factors of $\sin x$ affect my statement that the factor theorem

is 'really difficult'? Apart from details his proof is that which I was taught years ago—due, I believe, to Tannery (if not much older). And does Mr. Picken mean to imply that his proof is not 'really difficult.'?

(2) Mr Picken seems to me to follow Prof. Chrystal in a certain vagueness as to the distinction between a value for $x=1$ and a limit for $x=1$. The function $\frac{x^2-1}{x-1}$ has *no* value for $x=1$; for $x=1$, $\frac{x^2-1}{x-1}$ is strictly and absolutely meaningless. The fact that its limit for $x=1$ is 2 is entirely irrelevant. The functions $\frac{x^2-1}{x-1}$ and $x+1$ are different functions. They are equal when x is *not* equal to 1. Similarly the function $y=\frac{x}{x}$ is $=1$ when $x \neq 0$ and undefined for $x=0$. To calculate $f(x)$ for $x=0$ we must put $x=0$ in the expression of $f(x)$ and perform the arithmetical operations which the form of the function prescribes, and this we cannot do in this case.

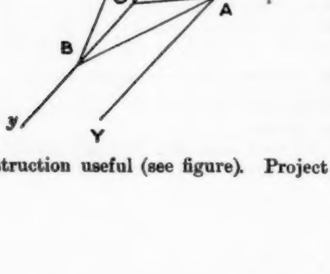
Whether Mr. Picken agrees with me here I cannot say. I lay stress on the point because his language is not quite clear. Thus he says (p. 330) that 'a function of x may have a value for a given value a of the argument, although the expression $f(x)$ fails to provide a value when a is substituted for it'—and I might quote other sentences which I cannot regard as entirely satisfactory. G. H. HARDY.

224. [M^1. 8. g.] *A curious imaginary curve.*

The curve $$(x+iy)^2=\lambda(x-iy)$$
is (i) a parabola, (ii) a rectangular hyperbola, and (iii) an equiangular spiral. The first two statements are evidently true. The polar equation is
$$r=\lambda e^{-3i\theta},$$
the equation of an equiangular spiral. The intrinsic equation is easily found to be $\rho=3is$.

It is instructive (i) to show that the equation of any curve which is both a parabola and a rectangular hyperbola can be put in the form given above, or in the form
$$(x+iy)^2=x \text{ (or } y),$$
and (ii) to determine the intrinsic equation directly from one of the latter forms of the Cartesian equation. G. H. HARDY.

225. [L^1. 1. a.] *The line at infinity, etc.*



Can anyone tell me of an English book which contains a clear and intelligible account of the 'line at infinity'? Such accounts as are contained in the ordinary books on Conics, or in Miss Scott's *Modern Analytical Geometry*, appear to me confusing in the highest degree.

Most undergraduates seem to believe that there really are points at infinity, and that they really do lie on a line, and that if you could get there you would find that $1=0$. The fault lies in the books, which persist in treating conventions as if they were sober statements of fact.

I have found the following construction useful (see figure). Project p in the plane $x+y+z=1$ into P in

the plane $x=1$. Taking axes AY, AZ, as shown, in the latter plane we find

$$Y=\frac{y}{x},\ Z=\frac{z}{x},$$

x, y, z being the Cartesian coordinates of p—or (what is the same thing) the areal coordinates of P in the plane $x+y+z=1$, referred to the triangle ABC—and Y, Z the Cartesian coordinates of P in the plane AYZ.

Thus if the locus of P is $f(1, Y, Z)=0$, that of p is $f(x, y, z)=0$.

G. H. HARDY.

226. [L[1]. 2. b.] *Morley's Problem.*

Let ab, bc, cd, de, ea, ac, bd, ce, da, eb be denoted by A, B, C, D, E, S, T, P, Q, R.

Let U, V be the conics $ABCDE$, $PQRST$, and X, Y those that touch $abcde$ and PQ, QR, RS, ST, TP respectively. Then the collineation transforms A, B, C, D, E, U, X into P, Q, R, S, T, V, Y respectively.

Now PCD, QDE, REA, SAB, TBC are triangles described about X, and each has two vertices on U. Hence in each case the third vertex lies on a fixed conic having a common self-polar triangle with X, U. This conic must be V, since it goes through P, Q, R, S, T.

Also PQR, QRS, RST, STP, TPQ are triangles inscribed in V, and two sides of each touch Y. Hence the third sides touch a fixed conic having a common self-polar triangle with V, Y. This conic must be X, since it touches a, b, c, d, e. Hence U, V, X, Y have a common self-polar triangle, which is therefore unchanged by the collineation.

This self-polar triangle must therefore be the fixed triangle of the collineation unless the collineation has the period two or three, that is, can be reduced by projection to a rotation through 180° or 120°. It seems evident that this does not generally happen, and in fact it never happens. It is not hard to work out the actual transformation when ABC is the triangle of reference, and the multipliers λ, μ, ν are then found to satisfy the condition $\Sigma\lambda\,.\,\Sigma 1/\lambda+1=0$. In the two cases mentioned we have respectively

$$\lambda^2=\mu^2=\nu^2 \quad\text{and}\quad \lambda^3=\mu^3=\nu^3,$$

which are inconsistent with this condition.

Since U, V may be written

$$\lambda^2x^2+\mu^2y^2+\nu^2z^2 \quad\text{and}\quad x^2+y^2+z^2,$$

it is easy to find the relation connecting their invariants, namely,

$$125\Delta^2\Delta'^2+10\Delta\Delta'\Theta\Theta'-4\Delta\Theta'^3-4\Delta'\Theta^3+\Theta^2\Theta'^2=0.$$

A. C. DIXON.

227. [I.] In a review (*Gazette*, March, 1906, vol. iii., pp. 302-304), I have stated that the name of the publisher of a certain pamphlet was not given. I now learn from a correspondent that the title page (missing in my copy) was:—"The power of the Continuum. Inaugural Dissertation zur Erlangung der Doktorwürde genehmigt von der hohen philosophischen Fakultät der Landesuniversität Rostock. Von Harold A. P. Pittard-Bullock aus Guernsey, Dezember, 1905. Referent: Professor Dr. Otto Staude, Druck von E. Ebering, Berlin, N.W." In view of the nature of the pamphlet, this must cause some surprise. PHILIP E. B. JOURDAIN.

228. [X. 6.] *A mechanical construction of curves representing the movement of a pendulum.*

Consider a right circular cylinder of radius r, and a point O on its axis. Let a sphere of radius b, and centre at any point C in the circular section through O, intersect the cylinder in a curve.

If z, θ are the cylindrical coordinates of any point P on the curve we have (fig. 1)

$$CP^2 = PM^2 + MC^2$$
$$= z^2 + c^2 + r^2 - 2cr\cos\theta, \text{ if } OC = c,$$
$$= z^2 + (c-r)^2 + 4cr\sin^2\frac{\theta}{2},$$
$$\text{or } z^2 = 4rc\left(\kappa^2 - \sin^2\frac{\theta}{2}\right), \text{ if } \kappa^2 = \frac{b^2 - \overline{c-r}^2}{4rc}.$$

Comparing this with the energy equation of a simple pendulum, length l, oscillating through an angle α on either side of the vertical, namely,

$$(l\dot{\theta})^2 = 2gl\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right),$$

we see that if $c = l$ and $\kappa = \sin\frac{\alpha}{2}$, then z represents $l\dot{\theta}$, that is $\dot{\theta}$, on a certain scale.

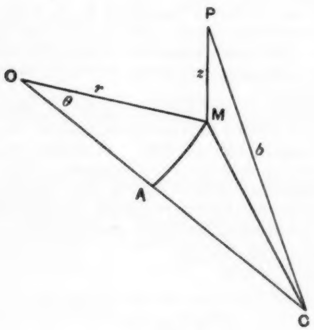


FIG. 1.

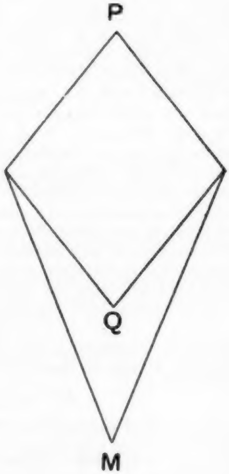


FIG. 2.

A paper may be wrapped on the cylinder and the curve described with an ordinary pair of knee-jointed compasses in the way a circle is usually drawn, the "centre" being at a distance l from the axis of the cylinder and the "radius" b being given by the equation

$$b^2 = (l-r)^2 + 4rl\sin^2\frac{\alpha}{2}$$
$$= l^2 + r^2 - 2rl\cos\alpha,$$

or by the fact that $CM = b$ when $COM = \alpha$.

If the paper is unrolled into a plane the equation becomes (referred to axes corresponding to $z = 0$ and $\theta = 0$)

$$z^2 = 4rc\left(\kappa^2 - \sin^2\frac{x}{2r}\right).$$

The plane curve is a graph of θ and $\dot{\theta}$. It is easily shewn that the subnormal represents the angular acceleration $\ddot{\theta}$.

If $\kappa^2=1$ or $b=l+r$ we obtain a cosine curve. If $\kappa^2>1$ the curve represents the movement of a particle making complete revolutions in a vertical circle under gravity.

A graph of $\frac{dt}{d\theta}$ and θ may be obtained in part by using Hart's or Peaucellier's inverter on the cylindrical surface. Causing M (figs. 1 and 2) to describe the circular section and constraining P to describe the curve aforesaid by a link CP, we obtain a Q curve, and since $MP.MQ$ is constant, MQ represents $\frac{dt}{d\theta}$, and the area of any part of the Q curve represents the time of the corresponding pendulum movement. Obviously only a portion of the Q curve can be actually described mechanically. In general the curve $f\left(2r\sin\frac{x}{2r}, z\right)=0$, can be obtained by adapting to the inside or outside of the cylindrical surface any construction which will draw the curve $f(x, z)=0$ on a plane.

C. S. JACKSON.

229. [V.] *Contracted Methods.*

I am sorry to see so keen an enthusiast on practical education as Mr. Godfrey objecting to contracted methods of multiplication and division, and to see that his objections are endorsed by Mr. Budden. Mr. C. S. Jackson has been in correspondence with me on the subject, and at his suggestion I venture to draw attention to a mode of working which seems to me to greatly assist quickness and accuracy in the use of the contracted method.

What I advocate is that, when contraction begins, each digit of the multiplier should, as it is about to be used, be placed immediately over the digit of the multiplicand with which that line of multiplication is to begin: then, as line after line of the partial products is written, the digits of the multiplicand which are still uncovered are those which are still in active use. By this means, even if the work is interrupted, the computer knows exactly where he has got to, and can resume his work without risk, especially if he either ticks or cancels the superscript digit as soon as it has done its work. The same applies in division, the digits of the quotient being, when contraction begins, placed over the digits of the divisor with which they commence. Or, in this case, they may be placed under, instead of over, if preferred.

Again, in extracting a square root to many figures, as soon as division begins, the operating digits can be placed under the proper figures of the divisor in the same way.

This method of 'figured' multiplicand, or 'figured' divisor, prevents any risk of making a partial product 10 times too great or 10 times too small, a mistake which I found often occurred when no such precaution was taken.

With regard to numerical slips in forming the products, these can be guarded against by trying over each line as soon as it is written, by mentally dividing it by the digit by whose means it was obtained; and, in multiplication, when the partial products are added together they should be added twice, once upwards, and once downwards, as Bank clerks do in adding money columns.

With regard to fixing the decimal point, the plan I prefer is to arrange, at commencing, for the operator (whether multiplier or divisor) to have its highest figure in the units place, so that the decimal point in the result is exactly where it would be in simple division by a single units figure.

Thus $\cdot0075867\times23\cdot58$

would be re-arranged as $\cdot075867\times2\cdot358$

and $\cdot0075867\div23\cdot58$

$=\cdot00075867\div2\cdot358$,

i.e. 23·58 is regarded as 10 times 2·358, and the operation by 10 is performed first.

[Another plan which many practical computers prefer is to perform the work without reference to decimals, and then fix the 'point' by marginal work with the highest figures.]

Lastly, when a product is to be true to so many figures, or so many decimal places, I put the unit (*i.e.* highest) figure of the multiplier under the digit of the multiplicand with which it is to commence, working with one more figure than asked for.

Some examples are appended by way of illustration.

I should be glad if these personal statements of method succeed in drawing expressions of criticism or assent, either in the *Gazette* or at the General Meeting of the Association, as the subject, though elementary, is one of considerable practical importance and interest.

Ex. 1. *To find the square of* 3·14159265 *to* 4 *decimal places.*

```
89 5141
 3·14159265
        3·14159265
------------------
 9·42478
   31416
   12566
     314
     157
      28
       1
------------------
 9·8696
```

Ex. 2. *Evaluate* ·0075867 ÷ 23·58 *to* 4 *significant figures.*

```
2·358 ) ·00075867 (·0003218 (–)
-----
  1       7074
          ----
           5127
           4716
           ----
            411
            236
            ---
            175
```

With regard to the question of writing down only the remainders, and not the partial products, my own feeling is that with a short divisor like the above it would have been just as easy to omit the products, but with long divisors, especially if a long quotient is wanted, it is better to insert the products, as it is so much easier to check the work, and moreover much time is saved if it should happen that digits in the quotient are repeated. I don't think contractions should be made into a 'fetish,' to be used whether felt to be desirable or not. At the same time, unless contractions are practised, they *cannot* be used, so for school work one would as a rule in senior forms cultivate the method of writing remainders only.

Ex. 3. *Find the square root of* 37·56361 *to* 6 *decimal places.*

```
          | 37·563610 (6·128916 (–)
      121 |  1 56
     1222 |    3536
    12248 |    109210
  12256,9 |      112260
  ------- |
     19   |        1948
          |         722
```

Note.—It has been pointed out to me by a colleague that one of the chief objections to contracted methods is that it is difficult or impossible to teach small boys to use them properly, and that it is needless to teach them to older boys, because then they are able to use logarithms, which are simpler and quicker. There *seems* to be force in this argument, and yet I think the objection is not a valid one, for it must be educationally and practically wrong to leave a boy unable to deal neatly with masses of figures out of which he requires to obtain a result to a moderate degree of accuracy, unless he can fly to a logarithm book. I should think that a boy who is advanced enough to use logarithms intelligently, should also be able in simple cases to dispense with them without having to drag in a whole host of useless figures.

I would like to say one word on the subject of logarithms.

In solving such an equation as

$$x=\frac{3{\cdot}142\times 73{\cdot}28\times 5{\cdot}923}{48{\cdot}34\times 39{\cdot}67}$$

beginners nearly always write:

$$\begin{aligned}\therefore\ \log x &= \log(3{\cdot}142)+\log(73{\cdot}28)+\text{etc.}\\ &= {\cdot}4972+1{\cdot}8650+{\cdot}7725-1{\cdot}6843-1{\cdot}5985\\ &= \bar{1}{\cdot}8519,\\ \therefore\ x &= 0{\cdot}7110,\end{aligned}$$

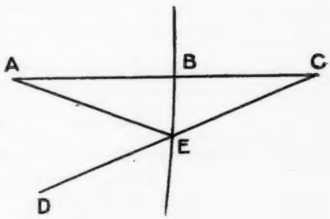
instead of doing the whole work in *columns*, thus:

$$\begin{array}{rlrl} \log 3{\cdot}142 = & 0{\cdot}4972 & \log 48{\cdot}34 = & 1{\cdot}6843 \\ \log 73{\cdot}28 = & 1{\cdot}8650 & \log 39{\cdot}67 = & \underline{1{\cdot}5985} \\ \log 5{\cdot}923 = & \underline{0{\cdot}7725} & & 3{\cdot}2828 \\ \log \text{Num.} = & 3{\cdot}1347 & & \\ \log \text{Den.} = & \underline{3{\cdot}2828} & & \\ \log x = & \bar{1}{\cdot}8519 & \therefore\ x = & 0{\cdot}7110 \end{array}$$

It cannot be too much insisted on that for numerical calculations the column method is the proper one, the method of writing out work in *line* form being appropriate to Algebra, but not to Arithmetic. It will usually be found that the student who writes out the above in lines has on some odd piece of paper put it into columns: indeed, it would simply be inviting disaster if he did not. ALFRED LODGE.

230. [K.] *To prove* Euc. I. 27 *without reversing the figure on itself or using superposition and assuming nothing but* Euc. I. 4 *and* 13.

Let a line cut two lines AB, DE at B and E so that $A\hat{B}E=B\hat{E}C$ and consequently $C\hat{B}E=B\hat{E}D$ (I. 13).



If possible let AB and DE meet at C.

Make BA equal to CE and join AE.

Then in triangles BEC, ABE

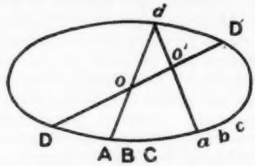
$AB=CE$, BE is common,

$A\hat{B}E=B\hat{E}C$;

$$\therefore\ A\hat{E}B=E\hat{B}C=B\hat{E}D, \text{ which is absurd.}$$

$\therefore$ AB and DE do not meet towards B and E, and similarly not towards A and D. C. A. RUMSEY.

231. [K. 7. e.] *Given two collinear sets of three points A, B, C and a, b, c find α, β, γ in involution with both sets.*



If the points were on a conic or circle the construction would be as follows.

Let DD' be the homographic axis of the ranges determining the pairs (A, a), (B, b), (C, c).

Take a point α on the curve and let αA, αa cut DD' in O and O'.

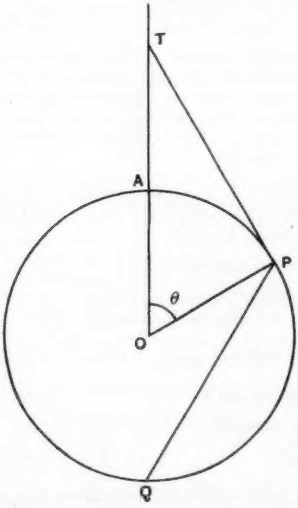
If now α, A, a be supposed to move on the conic so that αA and αa always pass through two fixed points O, O', the successive positions of α and A form pairs in involution, and so also do those of α and a.

Also A and a describe homographic ranges of which D and D' are double points and of which the original positions of A and a are conjugates, that is the homographic ranges determined by the three given pairs. Thus BO, bO' and CO, cO' intersect in the conic, and if these points of intersection are β, γ, then α, β, γ are the required set.

The extension to the case in which ABC, abc are collinear is obvious.

C. A. Rumsey.

232. [K. 7. a. α.] *A particle, attached to a fixed point by a string, moves in a vertical circle. If the string slackens, find where the particle meets the circle again.*



O is the centre of the circle; the particle leaves the circle at P. m = the mass of the particle; OA is a vertical radius. $\angle AOP = \theta$. v = the velocity at P.

Let ρ be the radius of curvature of the parabola in which the particle commences to move at P, and let a be the radius of the circle.

$$\therefore \frac{mv^2}{a}=mg\cos\theta,$$

$$\frac{mv^2}{\rho}=mg\cos\theta;$$

$$\therefore \rho=a.$$

$\therefore$ the circle is the circle of curvature to the parabola at P.

Hence if the particle meets the circle again at Q, PQ is equally inclined to the vertical with PT the tangent at P. P. SCOONES.

233. [D. 6. b.] The following modification of Stolz and Gmeiner's method (which Mr. Hardy numbers (iv)) was suggested by Mr. M'Cleland, one of my pupils, and appears to me simple and satisfactory.

Let $\exp(ix)=u+iv,$
then $\exp(-ix)=u-iv,$ $\Big\}$ x being real.

Then $$u^2+v^2=\exp(ix).\exp(-ix)=1.$$

$\therefore$ $\exp ix$ has modulus unity, *i.e.* is represented by a point on the circle radius unity, centre origin ;

$$\therefore \exp(ix)=\cos\phi+i\sin\phi, \text{ where } \phi \text{ is a real angle.}$$

Differentiating, $\exp(ix).idx=(-\sin\phi+i\cos\phi)d\phi$;

$$\therefore \frac{idx}{d\phi}=\frac{-\sin\phi+i\cos\phi}{\cos\phi+i\sin\phi}=i; \quad \therefore \frac{dx}{d\phi}=1;$$

$$\therefore x=\phi+\text{constant}=\phi+c, \text{ say}$$

(thus to each value of ϕ, a value of x corresponds, as well as conversely);

$$\therefore \cos\phi+i\sin\phi=\exp(i\phi+ic) \text{ for all real values of } \phi;$$

$$\therefore \cos(-c)+i\sin(-c)=\exp(0)=1;$$

$$\therefore c=2k\pi \text{ where } k \text{ is an integer};$$

$$\therefore \cos\phi+i\sin\phi=\exp i(\phi+2k\pi).$$

This proof has the advantage of avoiding the initial assumption in Mr. Godfrey's way of writing the proof (viz. that y can be found so that $\cos x+i\sin x=\exp y$). C. O. TUCKEY.

234. [L[1]. 17. e.] *On a Certain Double Envelope.*

How often in Conics must poor despised Geometry come to the rescue of Analysis—to test, interpret, and illumine results of themselves stale, flat, and unprofitable !

A striking illustration of the superiority of geometrical treatment is to be found in the consideration of the following attractive question due to Mr. C. E. Youngman, of Cheltenham College :

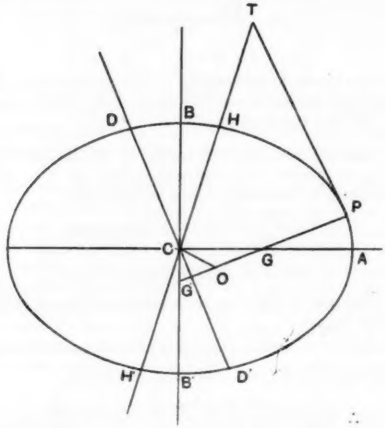
"Prove that similar ellipses having a common diameter envelope a pair of circles." [*Educational Times*, October 1906, No. 16087.]

(I.) The analyst sets to work somewhat in this fashion :

Assume $x^2-d^2+2\lambda xy+\beta y^2=0$ as the equation to the ellipse, where λ, β are variable with the condition $\sqrt{(\lambda^2-\beta)/(1+\beta)}=\text{const.}$ Though never in any actual difficulty he is certainly completely in the dark from the outset until the end. Finally perseverance is rewarded by some such result for the envelope as $(x^2+y^2-d^2)^2=\kappa^2y^2$. Then with a sigh of relief he feels he has achieved great things ; and that the question, as far as he is concerned, is completely disposed of.

(II.) The geometrician finds another and a better way:

Let the normal at a point P of an ellipse meet the axes in G, G' respectively: so that $PG=(b/a).CD$ and $PG'=(a/b).CD$, where CD is conjugate to CP. Also let HCH' be equally inclined with DCD' to the minor axis BCB'.



Then since $THCH'$ and the tangent TP are equally inclined to either axis a circle can be described passing through H, H' and touching the ellipse at P. Its centre will be at O, the middle point of GG', because OC is perpendicular to CH (angle $OCG=OGC=BCD=BCH$). Its radius

$$OP=\tfrac{1}{2}(PG'+PG) \text{ and } OC=OG=OG'=\tfrac{1}{2}(PG'-PG).$$

Moreover OP and OC both vary (since the excentricity is given), as CD, that is as CH, which is known. The circle is therefore invariable for all similar ellipses having a given diameter HCH'.

By symmetry the ellipses also touch a second invariable circle, the image of the first in HCH'. R. F. Davis.

235. [V. 4. b. β.] *Graphical solution of a biquadratic.*

Any biquadratic $ax^4+bx^3+cx^2+dx+e=0$,

by the substitution $x=\tan\frac{\theta}{2}$, may be transformed into

$$A\sin 2\theta+B\cos 2\theta+C\sin\theta+D\cos\theta+E=0;$$

when $$\frac{A}{d-b}=\frac{B}{a+e-c}=\frac{C}{2b+2d}=\frac{D}{4c-4a}=\frac{E}{3a+3e-c}.$$

Draw the limaçon $r=-C\sin\theta-D\cos\theta-E$,

and the quatrefoil $r=A\sin 2\theta+B\cos 2\theta$,

they will intersect in four points at most.

Measure with a protractor the angles θ of their intersections; find from the tables the values of $\tan\frac{\theta}{2}$; these are the roots of the biquadratic.

Any number of points on these curves is easily obtained graphically.

For the quatrefoil: determine graphically $F=\sqrt{(A^2+B^2)}$; draw a circle, centre O, radius $2F$; take $X'OX$, $Y'OY$ two fixed diameters at right angles

and $Q'OQ$, $R'OR$ any other pair of diameters at right angles; draw QN, $Q'N'$ perpendicular to OY; NP, $N'P'$ perpendicular to OR, determining two points P, P' on the quatrefoil.

For the limaçon: measure $-C$, $-D$ along the axes; complete the rectangle; with the intersection of its diagonals as centre, draw a circle passing through the origin; along radii vectores from the origin measure lengths E from the circle, both within and without, determining two points on the limaçon.

Draw one, or both, of these curves on tracing paper; place the origins one on the other so that the x-axis of the quatrefoil is inclined at an angle $-\frac{1}{2}\tan^{-1}B/A$, which has been previously determined graphically, to the x-axis of the limaçon.

Prick off the points of intersection. W. H. H. HUDSON.

236. [L_2. 4. a.] *The radius of the general right circular cylinder and the equation of its axis.*

Let the equation of the right circular cylinder be

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0. \quad\text{.........(1)}$$

Then $ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d$

$$+(lx+my+nz+p)^2=0 \quad\text{............(2)}$$

is the system of conicoids having contact with (1) round the curves of section by the planes $lx+my+nz+p=0$. Now if (1) is a right circular cylinder and the plane of section is perpendicular to its axis, in which case l, m, n will be proportional to the direction cosines of that axis, one of the conicoids having contact with (1) round the curve of section by that plane will be a sphere; and whatever plane is taken perpendicular to the axis the radius of the corresponding sphere will be constant and equal to the radius of the cylinder. Now if (2) is a sphere $a+l^2=b+m^2=c+n^2=k$ say, $mn+f=0$, $nl+g=0$, $lm+h=0$. Also the radius of the sphere is

$$\sqrt{\left(\frac{u+lp}{k}\right)^2+\left(\frac{v+mp}{k}\right)^2+\left(\frac{w+np}{k}\right)^2-\left(\frac{d+p^2}{k}\right)}.$$

Now the quantities l, m, n are not only proportional to the direction cosines of the axis, they are actual constants wherever the perpendicular plane is taken, as can be seen by the solutions to the above equations of condition found in the previous note. Hence also k is constant and the expression under the radical is a quadratic in the single variable p.

But since we have proved that this expression is a constant, being the square of the radius of the cylinder, the coefficients of p and of p^2 must each be zero. Thus the radius of the cylinder is

$$\sqrt{\frac{u^2+v^2+w^2-kd}{k^2}},$$

where $k=a-\frac{gh}{f}$, when f, g, h are all non-zero,

where $k=a$, when $g=0$ and $h=0$,

and where, when $f=0$, $g=0$, $h=0$, k is equal to the two equal quantities among the quantities a, b, c; as can be seen by the work in the previous note.

Again the axis is the locus of the centres of the spheres mentioned above. But the centre of the sphere (2) is x, y, z, where

$$x=-(u+lp)/k, \quad y=-(v+mp)/k, \quad z=-(w+np)/k.$$

Thus $\frac{x+u/k}{l}=\frac{y+v/k}{m}=\frac{z+w/k}{n}$ is the equation of the axis. Thus getting l, m, n from the work of the first note we have for the equation of the axis

$$\frac{x+u/k}{1/f}=\frac{y+v/k}{1/g}=\frac{z+w/k}{1/h},$$

when fgh are all non-zero.

$$\frac{x+u/k}{0}=\frac{y+v/k}{\sqrt{a-b}}=\frac{z+w/k}{\sqrt{a-c}},$$

when $g=0, h=0$, the signs given to the radicals being such as to make $\sqrt{a-b}$, $\sqrt{a-c}$ have the opposite sign to f.

$$\frac{x+u/k}{1}=\frac{y+v/k}{0}=\frac{z+w/k}{0},$$

when $f=0, g=0, h=0$, and the two equal quantities among a, b, c are b and c.

H. L. TRACHTENBERG.

REVIEW.

Geometrical and Projective Sketching. (Mathematical Drawing.) By C. B. M'ELWEE.

An interesting paper on the above subject was given before the members of the Royal Drawing Society on April 20th, 1906, by Mr. Charles B. M'Elwee, Royal Indian Engineering College, Coopers Hill. Blackboard demonstrations, models, etc., were used to elucidate the subject, but the difficulties attending the reproduction of these diagrams, etc., makes it quite impossible to give a clear and intelligible description of the system he advocated. However, some interesting points were touched upon by Mr. M'Elwee, and these may prove of interest to those who are in any way connected with the teaching of the subjects in question, and it is these alone which we offer for consideration. The lecturer having first pointed out the difficulties students have in understanding orthographic projections "in view of the fact that they do not represent the object as seen by the eye," with other difficulties of a technical nature, suggested that these may be overcome by studying pictorial representations of objects and conjointly the same when orthographically represented, "for by so doing you can accustom the mind to realize the appearance and identity of the object so represented." He proceeded to suggest that the difficulties of Perspective projection might be overcome, to some extent, by practice in drawing objects by Freehand, though acknowledging that this would only supply a limited accuracy when compared with the precision of mathematical instruments, properly used. But the lecturer rightly objected to a constant use of these mechanical aids. He put the question, "Are the introduction and use of Isometric and Oblique projections, as methods of pictorial expression, educational?" and we think well answered it by saying, "Artistically they may tend to warp and stultify any development of the power of observation and pictorial expression to be found in the pupil; and scientifically they may instil principles which, to some, appear to violate the mathematical laws involved in the construction of a true perspective projection." The difficulties besetting students who attempt a perspective drawing without some knowledge of Solid Geometry, and the practical impossibility of success, was another point which will appeal to those who have occasion to teach the subject. The method advocated by Mr. M'Elwee is of an ingenious nature, even if it seems to us lacking some measure of directness. The lecturer himself, in conclusion, appears to recognize this, for he remarks: "I might point out, this method of perspective or geometrical and projective sketching is not intended to supersede or supplant any pictorial method of teaching drawing; indeed, I must impress upon you that a certain amount of skill in observation drawing is necessary before the pupil can derive any benefit from its use."

R. B. P.

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